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Planck-Mass-Rotons Cold Dark Matter Hypothesis

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Sakharov's conjecture that the vacuum is densely occupied with Planck-mass maximons is taken as a model to explain the missing mass as rotons of a superfluid made up from the Planck-mass maximons. Because rotons require a finite excitation energy, they not only can account for the missing mass but, in addition, can mimic a small, positive cosmological constant. According to Sakharov, the large vacuum energy of the Planck-mass maximons is compensated by "ghost particles." In the proposed superfluid vacuum model, we assume that the compensation is done by a large, negative cosmological constant instead.

1. INTRODUCTION

There is growing evidence that the observed large-scale structure of the universe can best be described by the combination of a cold dark matter component and a small, fine-tuned cosmological constant. Because the required small cosmological constant is difficult to reconcile with our present understanding of elementary particle physics, Weinberg (1992) has entertained the "anthropic principle" for an explanation of this peculiarity.

Quantum field theory has to make the assumption that the energy-momentum tensor of the quantum fluctuations of the vacuum are compensated by a gravitational field action $S(0)$, where

$$S(R) = -\frac{c^3}{16\pi G}\int R\sqrt{-g}\, d\Omega \tag{1.1}$$

is the Einstein–Hilbert gravitational field action (Landau and Lifshitz, 1971). It was suggested by Zel'dovich (1967) that a small disturbance of this equilibrium could lead to a finite value of the cosmological constant. Following this suggestion, it was argued by Sakharov (1968) that by expanding the

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gravitational field Lagrangian in a series of powers of the curvature (A, B are numerical constants of order unity)

$$\mathscr{L}(R) = \mathscr{L}(0) + A\hbar R \int k\,dk + B\hbar R^2 \int \frac{dk}{k} + \cdots \tag{1.2}$$

one can assign the first term in (1.2) to the cosmological constant and the second term to the action (1.1). The third- and higher-order terms lead to nonlinear departures from Einstein's gravitational field equations of no interest here. Comparing the second term with (1.1), one has

$$G = \frac{c^3}{16\pi A\hbar \int_0^{k_0} k\,dk} \tag{1.3}$$

where the cutoff wave number is determined by the numerical value of G:

$$\int_0^{k_0} k\,dk \sim k_0^2 \simeq c^3/\hbar G \tag{1.4}$$

For the cutoff wave number k_0, one thus finds that $k_0 = 1/r_P$, where $r_P = (\hbar G/c^3)^{1/2} \simeq 10^{-33}$ cm is the Planck length. Particles belonging to the wave number $k_0 \simeq 10^{33}\ \text{cm}^{-1}$, with a mass of $m_P = (\hbar c/G)^{1/2} \simeq 10^{-5}$ g, have been called "maximons." They have been proposed by various authors as the heaviest hypothetical particles in nature. To describe the vacuum in a way consistent with the Lagrangian (1.2), Sakharov suggests as a simple model a vacuum densely filled with maximons, one maximon per Planck-length volume. To cancel the otherwise huge vacuum mass density ($c^5/\hbar G^2 \sim 10^{95}$ g/cm^3), he assumes that there is an equal number of compensating "ghost particles." Sakharov's model is consistent with the qualitative consequences of quantum gravity arrived at by Wheeler (1968) and Hawking (1978).

To simplify Sakharov's model, we propose that the compensating ghost particles are replaced by a large, negative cosmological constant of the order $\sim -1/r_P^2$. It, too, can cancel the large vacuum energy of the maximons. We furthermore propose that the maximons can be described by a zero-temperature superfluid.[2]

Assuming that the maximons have contact-type interactions, they can be described by the following nonlinear Schrödinger equation known from the theory of superfluidity:

[2]The idea that the vacuum is a superfluid but instead composed of fermion–antifermion pairs was suggested some time ago by Sinha *et al.* (1976a,b; Sinha and Sudarshan, 1978).

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_P} \nabla^2 \psi + f^2 \psi^* \psi^2 \tag{1.5}$$

In (1.5), f^2 is a coupling constant, the strength of which determines the assumed local interaction between the maximons. As will be shown, this vacuum model can explain the missing mass and a fine-tuned cosmological constant to result from the rotons of this superfluid.

In its hydrodynamic form, (1.5) becomes

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= -\frac{1}{m_P} \nabla(V + Q) \\ \frac{\partial n}{\partial t} &+ \nabla \cdot (n\mathbf{v}) = 0 \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} n &= \psi^* \psi \\ n\mathbf{v} &= -\frac{i\hbar}{2m_P} [\psi^* \nabla \psi - \psi \nabla \psi^*] \\ V &= f^2 n \\ Q &= -\frac{\hbar^2}{2m_P} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \end{aligned} \tag{1.7}$$

Introducing the velocity potential ϕ

$$\mathbf{v} = -\nabla \phi \tag{1.8}$$

and the function

$$W(n) = \frac{1}{nm_P} \int (V + Q)\, dn \tag{1.9}$$

we can derive the hydrodynamic equations (1.6) from the Lagrange density

$$\mathcal{L}_1 = nm_P \left[\dot{\phi} - \frac{1}{2} (\nabla \phi)^2 - W(n) \right] \tag{1.10}$$

Variation with regard to n leads to Bernoulli's equation

$$\dot{\phi} - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{m_P} (V + Q) = 0 \tag{1.11}$$

Euler's equation (1.6) is obtained by taking the gradient of (1.11). Variation of (1.10) with regard to ϕ leads to the equation of continuity:

$$-\dot{n} + \mathrm{div}(n\nabla\phi) = 0 \tag{1.12}$$

2. PHONONS AND ROTONS NEAR THE PLANCK SCALE

Keeping in $\mathcal{L}_1$ only terms quadratic in n, ϕ, or the product of n and ϕ, one obtains the approximate Lagrange density

$$\mathcal{L}_2 = n\dot{\phi} - \frac{n_0}{2}(\nabla\phi)^2 - \frac{1}{2}\frac{f^2 n^2}{m_P} \tag{2.1}$$

where $n_0 = 1/r_P^3$ is the number density of the maximons, with $n << n_0$ a small disturbance imposed on n_0. Variation of (2.1) with regard to n and ϕ now leads to

$$\begin{aligned} \dot{\phi} - (f^2/m_P)n &= 0 \\ -\dot{n} + n_0\nabla^2\phi &= 0 \end{aligned} \tag{2.2}$$

from which, by elimination of n, one obtains the wave equation

$$-\frac{1}{c^2}\ddot{\phi} + \nabla^2\phi = 0 \tag{2.3}$$

with

$$c^2 = n_0 f^2/m_P \tag{2.4}$$

Requiring that the wave propagation velocity is equal the velocity of light, one finds from (2.4) that $f^2 = \hbar c r_P^2$ because of $n_0 = 1/r_P^3$ and $m_P r_P c = \hbar$.

The spectrum of the densely-packed maximons, therefore, shows the typical feature of the phonon spectrum known from the theory of superfluidity.

Besides phonons, the spectrum has rotons. They are located below the Planck energy $m_P c^2 = \hbar c/r_P$. Following Pitaevskii (1956), we can most easily derive the rotons from the Hamilton density

$$\mathcal{H} = \frac{n_0 m_P}{2} v^2 + \frac{m_P c^2}{2n_0} n^2 \tag{2.5}$$

With the expansions

$$\begin{aligned} n &= \sum n^k e^{i\mathbf{k}\cdot\mathbf{r}} \\ \mathbf{v} &= \sum \mathbf{v}^k e^{i\mathbf{k}\cdot\mathbf{r}} \end{aligned} \tag{2.6}$$

the linearized continuity equation

$$\dot{n} + n_0 \nabla \cdot \mathbf{v} = 0 \tag{2.7}$$

becomes

$$\dot{n}^k + i n_0 \mathbf{k} \cdot \mathbf{v}^k = 0 \tag{2.8}$$

hence

$$\begin{aligned} \mathcal{H} &= \sum \frac{m_P}{2n_0} \left(\frac{|\dot{n}^k|^2}{k^2} + c^2 |n^k|^2 \right) \\ &= \sum \frac{m_P}{2n_0 k^2} \left(|\dot{n}^k|^2 + \omega^2 |n^k|^2 \right) \end{aligned} \tag{2.9}$$

where $\omega = kc$. It follows that the assembly of maximons behaves like an assembly of harmonic oscillators of frequency ω. The potential energy of a harmonic oscillator in its ground state is $(1/4)\hbar\omega$, and one obtains from (2.9) for each mode

$$n_0 \frac{1}{4} \hbar\omega = \frac{m_P}{2n_0 k^2} \omega^2 |n^k|^2 \tag{2.10}$$

Putting $\omega = E/\hbar$, one thus finds

$$E = \frac{\hbar^2 k^2}{2m_P S(k)} \tag{2.11}$$

where

$$S(k) = |n^k|^2 / n_0^2 \tag{2.12}$$

is the Fourier transform of the liquid structure function of the densely packed maximons. The corresponding expressions for superfluid helium were derived by Feynman (1955). For $k << k_P$, where $k_P = m_P c/\hbar$, one has $S(k) = (1/2)k/k_P$, and for $k >> k_P$, $S(k) = 1$. For $k_0 \lesssim k_P$, $S(k)$ goes through a maximum, with E having a minimum. It is this minimum which is associated with the rotons. Without the assumed discrete structure of the maximons densely filling the vacuum, there would be no rotons.

If the energy of the maximons is exactly compensated by a large, negative cosmological constant, the total vacuum energy is zero, making the flatness parameter $\Omega = 1$.

3. PLANCK-MASS ROTONS AS COLD DARK MATTER

The medium composed of the densely packed maximons is, for temperatures small compared to the Planck temperature T_P ($kT_P = m_P c^2$), superfluid.

Under these conditions, one has the typical phonon–roton energy spectrum shown in Fig. 1. For sufficiently low energies, it is described by a spectrum consisting of phonons, as in a solid. But for energies near the upper cutoff, which in our model is the Planck energy $m_P c^2$, the spectrum has a dip. Quasiparticles described by this part of the energy spectrum are called rotons, and since at the minimum of the dip $dE/dk = 0$, the rotons must have a small velocity. The height of the minimum of the dip can be viewed as an energy gap Δ, which by order of magnitude is $\Delta \sim m_P c^2$. With the width Δk of the dip of the order $\sim r_P^{-1}$, the excited state of rotons behaves like a fluid of free particles, with each roton mass approximately equal to m_P and having the gravitational charge $\sqrt{G} m_P$. A fluid composed of rotons has a finite pressure even if the velocity of the rotons vanishes. Because a finite pressure can mimic a cosmological constant, a roton fluid can mimic a cold gas composed of low-velocity particles, with a superimposed cosmological constant.

Near the minimum at $k = k_0$, (2.1) has the form

$$E = \hbar\omega = \Delta + \frac{\hbar^2 (k - k_0)^2}{2m_r} \tag{3.1}$$

where $m_r \lesssim m_P$ is the roton mass. The rotons to the right of the minimum at $k = k_0$ are called the R^+ rotons, and those to the left the R^- rotons. The velocity of the rotons is the group velocity

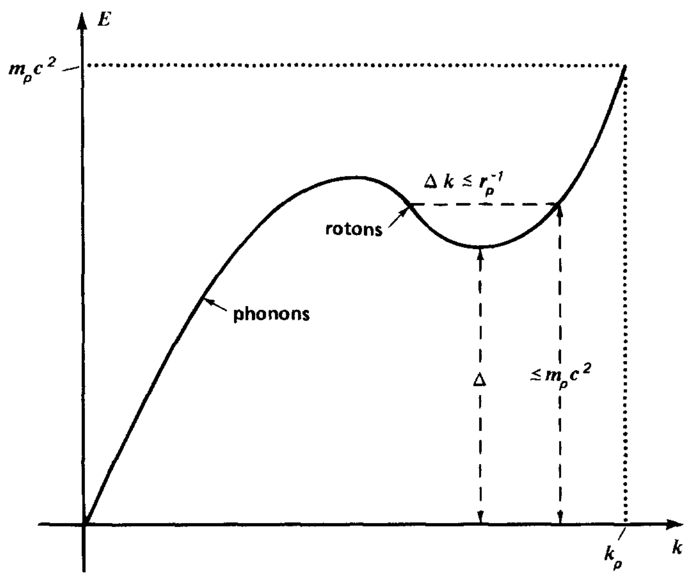

Fig. 1. The phonon–roton energy spectrum of the maximon fluid.

$$v = \frac{d\omega}{dk} = \frac{\hbar(k - k_0)}{m_r} \tag{3.2}$$

and their momentum is

$$p = p_0 + mv \tag{3.3}$$

where $p_0 = \hbar k_0$ is the momentum at $k = k_0$ and where $v = 0$.

Davis *et al.* (1985, 1988) have shown that a very good match for the observed matter distribution in the universe is obtained if the contribution of a cosmological constant (Λ) to Ω is $\Omega_\Lambda \simeq 0.8 \pm 0.1$, and that of cold dark matter (CDM) is $\Omega_{CDM} \simeq 0.2 \pm 0.1$, making $\Omega_\Lambda + \Omega_{CDM} = 1$. This twofold requirement, needed to reach $\Omega = 1$, is the so-called Ω-problem (Kolb and Turner, 1990). It is the strange property of the rotons that they are able to account for both contributions, making them a unique candidate to account for the missing mass.

According to (3.1) and (3.2), the total energy of a roton is

$$E = \Delta + \frac{1}{2} m_r v^2 \tag{3.4}$$

For relativistic roton energies, it can be written as follows:

$$E = (\Delta - m_r c^2) + \frac{m_r c^2}{(1 - v^2/c^2)^{1/2}} \tag{3.5}$$

with the first term making the contribution to Ω_Λ and the second one that to Ω_{CDM}. To obtain values for Δ and m_r, needed to estimate the contributions Ω_Λ and Ω_{CDM} to Ω, we assume that the phonon–roton spectrum is universal. Under this assumption, we can use the phonon–roton spectrum obtained from measurements in superfluid liquid helium by Henshaw and Woods (1961). According to these measurements, one has $\Delta \simeq 0.52 m_P c^2$ (equating the Debye energy with the Planck energy $m_P c^2$) and $m_r \simeq 0.16 m_P$ (equating the helium mass with the Planck mass m_P). Inserting these values into (3.5), one finds for the energy of a roton

$$\frac{E}{m_P c^2} \simeq 0.36 + \frac{0.16}{(1 - v^2/c^2)^{1/2}} \tag{3.6}$$

From this expression, one immediately sees that $\Omega_\Lambda \simeq 0.7$ and $\Omega_{CDM} \simeq 0.3$, with $\Omega_\Lambda + \Omega_{CDM} = 1$. This is a surprisingly good agreement with the empirical values $\Omega_\Lambda \simeq 0.8 \pm 0.1$ and $\Omega_{CDM} \simeq 0.2 \pm 0.1$, to explain the matter distribution in the universe.[3]

[3] The values for Ω_Λ and Ω_{CDM} are actually somewhat smaller to make room for cold baryonic matter, having a fraction less than 0.1. The contribution coming from hot baryonic matter is much smaller still, possibly as small as $\sim 10^{-3}$.

A roton number density equal to $n_P \simeq 2 \times 10^{-25}$ cm^{-3}, with a roton mass $m_r \lesssim m_P \simeq 2.2 \times 10^{-5}$ g, would be sufficient to reach the critical density $\rho \simeq 4.5 \times 10^{-30}$ g/cm^3 needed to make $\Omega = 1$. At this number density the distance in between two rotons would be $n_P^{-1/3} \simeq 1700$ km. This low number density, combined with their weak gravitational interaction, would make a direct detection of the rotons very difficult.

4. THE ROTATION CURVES OF DISC GALAXIES

Because the mass of the rotons is large, the velocity of the rotons relative to the galaxies must be small. Rotons in the intergalactic space fall toward the center of galaxies, and because of their very small cross section, they thereafter flow radially out. If the mass density of the rotons at a large distance r_0 measured from the center of attraction is ρ_0 and their velocity at this distance is v_0, their mass density ρ at the distance r is

$$\rho = \rho_0 \left(\frac{r_0}{r}\right)^2 \left(\frac{v_0}{v}\right) \tag{4.1}$$

Because the missing nonbaryonic mass is about 10 times larger than the baryonic mass, we can assume that the value of ρ_0 is equal to the critical density for $\Omega = 1$.

With (4.1), the gravitational potential is obtained from Poisson's equation in spherical coordinates:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d\phi}{dr}\right] = 4\pi G\rho \tag{4.2}$$

To compute the velocity for the rotons falling into the gravitational potential, we have to consider the dynamic behavior obtained from their dispersion relation (3.1), with the roton velocity given by their group velocity $d\omega/dk$.

The equation of motion of the rotons falling into a centrally symmetric gravitational potential ϕ is

$$m_r \frac{d}{dt}\left(\frac{d\omega}{dk}\right) = -m_r \frac{d\phi}{dr} \tag{4.3}$$

With $d/dt = (dr/dt)\, d/dr = v\, d/dr = (d\omega/dk)\, d/dr$, one obtains from (4.3) by integration

$$(d\omega/dk)^2 = v^2 = -2\phi \tag{4.4}$$

valid for both the R^+ and R^- rotons, with ρ the gravitational potential at the distance r. In spite of their unusual dispersion relation, the rotons, like any other form of matter, obey the equivalence principle.

At the distance $r = r_0$, where $\phi = \phi_0$, the velocity $v = v_0$ shall be related to Hubble's law

$$v_0 = Hr_0 \tag{4.5}$$

where

$$H = (8\pi G\rho_0/3)^{1/2} \tag{4.6}$$

is the Hubble constant with ρ_0 set equal to the value for which $\Omega = 1$. From (4.4)–(4.6), one has

$$\phi_0 = -(4\pi/3)G\rho_0 r_0^2 \tag{4.7}$$

With (4.1) and (4.7), (4.2) becomes

$$\frac{d}{dr}\left[r^2 \frac{d\phi}{dr}\right] = -3\phi_0\left(\frac{\phi_0}{\phi}\right)^{1/2} \tag{4.8}$$

Putting $r/r_0 = x$, $\phi/\phi_0 = y$, we can write (4.8) as follows:

$$\frac{d}{dx}\left[x^2 \frac{dy}{dx}\right] = -\frac{3}{\sqrt{y}} \tag{4.9}$$

with the boundary condition $y = 1$ at $x = 1$. Introducing the new variable

$$u = 1 - 3 \ln x \tag{4.10}$$

we find that (4.9) becomes

$$\frac{d}{dx}\left[x \frac{dy}{du}\right] = \frac{1}{\sqrt{y}} \tag{4.11}$$

with $y = 1$ at $u = 1$. With the substitution (4.10), we can also write (4.11) as follows:

$$-3\frac{d^2y}{du^2} + \frac{dy}{du} = \frac{1}{\sqrt{y}} \tag{4.12}$$

by which the independent variable u has been eliminated.

For (4.12), we seek solutions of the form

$$y = \sum_n a_n u^n \tag{4.13}$$

For $x \to 0$, that is, for $r \to 0$, one has $u \to \infty$. Therefore, if $u \to \infty$, only the highest power n contributes to the solution of (4.12).

In the limit $u \to \infty$, d^2y/du^2 can therefore be neglected against dy/du, and one obtains

$$\lim_{u\to\infty} y = (3/2)^{2/3}u^{2/3} \tag{4.14}$$

Hence

$$\lim_{r\to 0} \phi/\phi_0 = \{{}^3\!/_2\,[1 - 3\ln(r/r_0)]\}^{2/3} \tag{4.15}$$

By inserting into (4.12) for d^2y/du^2 and $\sqrt{y}$ the asymptotic solution (4.14), one obtains the approximate differential equation

$$\frac{dy}{du} \simeq \left(\frac{2}{3}\right)^{1/3} [u^{-1/3} - u^{-4/3}] \tag{4.16}$$

For $u \to \infty$, it has the same asymptotic solution as the exact differential equation. It therefore describes the departure from this asymptotic solution. By integration of (4.16), one obtains the approximate solution:

$$y \simeq \left(\frac{3}{2}\right)^{2/3} u^{2/3}\left[1 + \frac{2}{u}\right] \tag{4.17}$$

Higher approximations obtained by this iteration procedure lead to solutions of the form

$$y \simeq \left(\frac{3}{2}\right)^{2/3} u^{2/3} \sum_{n=0}^{\infty} a_n u^{-n} \tag{4.18}$$

In applying these results to the rotation curves of disc galaxies where $r << r_0$, it suffices to take the first term in the expansion of (4.18), which is the asymptotic solution (4.14):

$$\frac{d\phi}{dr} = -\frac{2(3/2)^{2/3}}{[1 - 3\ln(r/r_0)]^{1/3}}\frac{\phi_0}{r} \tag{4.19}$$

If the gravitational force acting on the visible matter of the galaxy is mainly determined by the mass of the rotons, then $-d\phi/dr$ has to be equated with the centrifugal force per unit mass, V^2/r, where V is the azimuthal velocity of the visible matter. One therefore has

$$V = \frac{\text{const}}{[1 - 3\ln(r/r_0)]^{1/6}} \tag{4.20}$$

For $r << r_0$, this is

$$V \simeq \text{const} \cdot [\ln(r_0/r)^3]^{-1/6} \tag{4.21}$$

showing a weak logarithmic dependence of V on r, in qualitative agreement with the observed flat rotation curves.

With the help of (4.6) and (4.7), one obtains

$$V = \frac{(3/2)^{1/3} H r_0}{[1 - \ln(r/r_0)^3]^{1/6}} \tag{4.22}$$

With $H \simeq 150$ km/sec/10^6 light-years, $r_0 \simeq 10^6$ light-years, and $r \simeq 10^5$ light-years (typical radius of a disc galaxy), one finds $V \simeq 130$ km/sec, in good agreement with the observed rotation velocities.

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